

Configurational Entropy for Travelling Solitons in Lorentz and CPT Breaking Systems

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In this work we group three research topics apparently disconnected, namely solitons, Lorentz symmetry breaking and entropy. Following a recent work [Phys. Lett. B **713** (2012) 304], we show that it is possible to construct in the context of travelling wave solutions a configurational entropy measure in functional space, from the field configurations. Thus, we investigate the existence and properties of travelling solitons in Lorentz and *CPT* breaking scenarios for a class of models with two interacting scalar fields. Here, we obtain a complete set of exact solutions for the model studied which display both double and single-kink configurations. In fact, such models are very important in applications that include Bloch branes, Skyrmions, Yang-Mills, Q-balls, oscillons and various superstring-motivated theories. We find that the so-called Configurational Entropy (CE) for travelling solitons, which we name as travelling Configurational Entropy (TCE), shows that the best value of parameter responsible to break the Lorentz symmetry is one where the energy density is distributed equally around the origin. In this way, the information-theoretical measure of travelling solitons in Lorentz symmetry violation scenarios opens a new window to probe situations where the parameters responsible for breaking the symmetries are random. In this case, the TCE selects the best value.

Keywords: entropy, non-linearity, Lorentz violation, kinks

1. INTRODUCTION

The most fundamental symmetry of the standard model of particle physics is the Lorentz invariance, which has been very well verified in several experiments. However, the first possibility of the Lorentz symmetry breaking was announced by Kostelecky and Samuel [1]. They argued that superstring theories indicate that Lorentz symmetry should be violated at higher energies. After that seminal work, a great number of works regarding the Lorentz symmetry violation (LSV) have appeared in the literature. Nowadays, the breaking of the Lorentz symmetry is a prominent mechanism for the description of several problems and conflicts in many areas of physics, from astrophysical [2–6] to subatomic scales [7–13]. For example, it was shown in an inflationary scenario with LSV [14] that, using a scalar-vector-tensor theory with Lorentz violation, the exact Lorentz violation inflationary solutions are found without the presence of the inflaton potential.

An important investigation line about topological defects in the presence of LSV have been recently addressed in the literature [15–18]. In this context, it was shown by Barreto and collaborators [15] that the violation of Lorentz and CPT symmetries is responsible by the appearance of an asymmetry between defects and anti-defects. Thus, in that context the authors showed that an analogous investigation can be used to build string theory scenarios. Motivated by this result, a class of travelling solitons in Lorentz and CPT breaking systems was presented in Ref. [18], where the solutions present a critical behavior controlled by the choice of an arbitrary integration constant. In this case, the field configurations have been shown to allow the emergence of so-called superluminal solitons [19]. Another increasing interest in LSV arises in investigations on neutrinos [20], gravity [21], electrodynamics [22], acoustic black hole [23, 24], monopoles and vortices [25–31].

On the other hand, in 1948, in an apparently disconnected topic, Shannon [32] described what is called “A mathematical theory of communication”, which nowadays is known as “Information theory”. In that work, Shannon introduced a mathematical theory capable of solving the most fundamental problem of communication, namely, the information transmission either exactly or approximately.

The main purpose of the information theory presented in [32] was to introduce the concepts of entropy and mutual information, by using the viewpoint of communication theory. In this context, the entropy was defined as a measure of “uncertainty” or “randomness”

of a random phenomenon. Thus, if a little deal of information about a random variable is received, the uncertainty decreases accordingly. As a consequence, one can measure this reduction in the uncertainty, which can be related to the quantity of transmitted information. This quantity is the so-called mutual information. After that work, a vast number of communication systems have been widely analysed from the information theory viewpoint, where the various types of information transmission can be studied under a unified model.

Moreover, in a cosmological scenario, Bardeen, Carter and Hawking [33] have established the relationship between the laws of thermodynamics and black holes. Some years later, Wald [34], motivated by the connection between black-hole and thermodynamics, has precisely defined the entropy for a self-gravitating system which contains a black hole. The ideas applied in [34] follow those ones provided by information theory.

Finally, in a very recent work [35], the concept of entropy has been, once more, reintroduced in the literature. Notwithstanding, now with an approach capable of taking into account the dynamical and the informational contents of models with localized energy configurations. In that letter, using an analogy to the Shannon's information entropy, the Configurational Entropy (CE) was constructed. It can be applied to several nonlinear scalar field models featuring solutions with spatially-localized energy. As pointed out in [35], the CE can resolve situations where the energies of the configurations are degenerate. In this case, the configurational entropy can be used to select the best configuration. The approach presented in [35] have been used to study the non-equilibrium dynamics of spontaneous symmetry breaking [36], to obtain the stability bound for compact objects [37], to investigate the emergence of localized objects during inflationary preheating [38], and moreover to distinguish configurations with energy-degenerate spatial profiles as well [39].

Hence, in this work we shall construct a CE in functional space, which we name Travelling Configurational Entropy (TCE), to measure the information of travelling solitons. It is worth to remark that the TCE can be used to study any physical model with energy density localized described by a travelling variable. In this case, the entropic measure opens a new theoretical window to probe the ordered arrangement of structures such as topological defects [40], ferromagnetic materials [41], solids far from equilibrium [42], and cosmic string [43] likewise. As an application, we shall investigate classical field theories in the context of Lorentz symmetry breaking and *CPT* violation, which admit energy density localized solutions. The model [44–52] that we will analyze has two interacting scalar fields and

admits a variety of kink-like solutions. Such model has been shown in the literature to give rise to Bloch branes [53–55], electrical conductivity phenomena in superconductors [56], bags, junctions, and in addition networks of BPS and non-BPS defects [57, 58].

This paper is organized as follows: in Section 2 we present the model which is going to be analyzed and we find the classical field configurations associated to it. In Section 3 the TCE measure is defined and we compute the information-entropic for travelling solitons in Lorentz and *CPT* breaking systems. In Section 4 we present our conclusions and final remarks.

2. THE MODEL

In this section we investigate classical field theories in the context of Lorentz symmetry breaking and *CPT* violation. In this case, the framework to study Lorentz and *CPT* violation is the so-called Standard-Model Extension (SME). Thus, in this context, we consider a two-field model in $(1 + 1)$ dimensions, where the Lorentz breaking Lagrangian density generalizes some works in the literature. In our theory, the Lagrangian density contains both vector functions and tensor terms. At this point, it is important to remark that the vector functions, which have a dependence on the dynamical scalar fields, are responsible for the Lorentz symmetry breaking. On the other hand, the tensor term breaks the Lorentz and, eventually, the *CPT* symmetry. Our motivation to write down such Lagrangian density comes from the work presented by Potting [59], where scalar field-theoretic models were presented. Hence, our generalized Lorentz breaking Lagrangian density is described by

$$\mathcal{L} = \frac{1}{2} \eta_1^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} \eta_2^{\mu\nu} \partial_\mu \chi \partial_\nu \chi + H^\mu \partial_\mu \phi + J^\nu \partial_\nu \chi + \partial_\mu \phi K^{\mu\nu} \partial_\nu \chi - V(\phi, \chi), \quad (1)$$

where $\mu, \nu = 0, 1$; $V(\phi, \chi)$ denotes the self-interaction potential; H^μ and J^ν are arbitrary vector functions of the fields ϕ and χ . In this case, such vector functions are responsible for the breaking of the Lorentz symmetry. On the other hand, $K^{\mu\nu} = K^{\mu\nu}(\phi, \chi)$ is a tensor function which represents the source of both LSV and *CPT* breaking symmetry. Here $K^{\mu\nu}$ is represented by a 2×2 matrix written in the form

$$K^{\mu\nu} = \begin{pmatrix} K^{00}(\phi, \chi) & K^{01}(\phi, \chi) \\ K^{10}(\phi, \chi) & K^{11}(\phi, \chi) \end{pmatrix}. \quad (2)$$

At this point, it is important to remark that the above matrix has arbitrary elements. However, if this matrix is real, symmetric, and traceless, the CPT symmetry is kept [60, 61]. Recently, a great number of works using a similar process for break the Lorentz symmetry, with a tensor like $K^{\mu\nu}$, have been used in the literature, from microscope [62] to cosmological scales [23, 24].

Moreover, the two effective metrics in the Lagrangian density (1) can be thought of as being perturbations of the Minkowski metric $\eta^{\mu\nu}$:

$$\eta_1^{\mu\nu}(\phi, \chi) = \eta^{\mu\nu} + F^{\mu\nu}(\phi, \chi), \quad (3)$$

$$\eta_2^{\mu\nu}(\phi, \chi) = \eta^{\mu\nu} + G^{\mu\nu}(\phi, \chi), \quad (4)$$

for arbitrary tensors components $F^{\mu\nu}$ and $G^{\mu\nu}$ with norm much less than the unity. The motivation to include such perturbations in the metric comes from the fact that the coefficients for LSV cannot be removed from the Lagrangian density, by using variables or fields redefinitions. Thus, observable effects of the LSV can be detected in the above theory.

Now, from the Lagrangian density (1), the Euler-Lagrange equations of motion for the two scalar fields ϕ and χ are respectively provided by:

$$\begin{aligned} \eta_1^{\mu\nu} \partial_\mu \partial_\nu \phi + K^{\mu\nu} \partial_\mu \partial_\nu \chi + (\partial_\mu \eta_1^{\mu\nu}) \partial_\nu \phi + \partial_\mu H^\mu + (\partial_\mu K^{\mu\nu}) \partial_\nu \chi - \frac{1}{2} \eta_1^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \eta_2^{\mu\nu} \partial_\mu \chi \partial_\nu \chi \\ - H_\phi^\mu \partial_\mu \phi - J_\phi^\nu \partial_\nu \chi - \partial_\mu \phi K_\phi^{\mu\nu} \partial_\nu \chi + V_\phi = 0, \end{aligned} \quad (5)$$

and

$$\begin{aligned} \eta_2^{\mu\nu} \partial_\mu \partial_\nu \phi + K^{\mu\nu} \partial_\mu \partial_\nu \chi + (\partial_\nu \eta_2^{\mu\nu}) \partial_\mu \phi + \partial_\nu J^\nu + (\partial_\nu K^{\mu\nu}) \partial_\mu \phi - \frac{1}{2} \eta_1^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \eta_2^{\mu\nu} \partial_\mu \chi \partial_\nu \chi \\ - H_\chi^\mu \partial_\mu \phi - J_\chi^\nu \partial_\nu \chi - \partial_\mu \phi K_\chi^{\mu\nu} \partial_\nu \chi + V_\chi = 0, \end{aligned} \quad (6)$$

where $A_\phi \equiv \partial A / \partial \phi$ [$A_\chi \equiv \partial A / \partial \chi$], for any quantity A dependent on ϕ [χ]. It can be seen that the two above equations carry information about the LSV of the model, through the presence of the tensors $K^{\mu\nu}$, the $\eta_i^{\mu\nu}$ metrics, and the vector functions as well.

For the sake of simplicity, and with our loss of generality, let us suppose that

$$F^{\mu\nu}(\phi, \chi) = f^{\mu\nu} = const., \quad (7)$$

$$G^{\mu\nu}(\phi, \chi) = g^{\mu\nu} = const., \quad (8)$$

$$K^{\mu\nu}(\phi, \chi) = k^{\mu\nu} = const., \quad (9)$$

where, $f^{\mu\nu}$, $g^{\mu\nu}$, and $k^{\mu\nu}$ are given by the matrices

$$f^{\mu\nu} = \begin{pmatrix} f^{00} & f^{01} \\ f^{10} & f^{11} \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{01} \\ g^{10} & g^{11} \end{pmatrix}, \quad k^{\mu\nu} = \begin{pmatrix} k^{00} & k^{01} \\ k^{10} & k^{11} \end{pmatrix}. \quad (10)$$

In $(1+1)$ dimensions, Eqs. (5) and (6) can be respectively expressed as

$$\begin{aligned} \eta_1^{00} \ddot{\phi} + K^{00} \ddot{\phi} + \eta_1^{11} \phi'' + (\eta_1^{10} + \eta_1^{01}) \dot{\phi}'^{10} + K^{01} \dot{\chi}' - H_\phi^0 \dot{\phi} - J_\phi^0 \dot{\chi} - H_\phi^1 \phi' \\ - J_\phi^1 \chi' + \dot{H}^0 + H^{1'} + V_\phi = 0, \end{aligned} \quad (11)$$

$$\begin{aligned} \eta_2^{11} \chi''^{00} \ddot{\phi} + \eta_2^{00} \ddot{\chi} + (\eta_2^{01} + \eta_2^{10}) \dot{\chi}'^{10} + K^{01} \dot{\phi}' - H_\chi^0 \dot{\phi} - J_\chi^0 \dot{\chi} - H_\chi^1 \phi' \\ - J_\chi^1 \phi' + \dot{J}^0 + J^{1'} + V_\chi = 0, \end{aligned} \quad (12)$$

where the prime [dot] stands for the derivative with respect to the space [time] dimension.

In general, as a consequence of the model studied in this work, we cannot analytically solve the above differential equations. However, one can still consider an interesting case for the field configurations, where travelling wave solutions are searched for. Travelling waves configurations have an important impact when boundary states for D-branes and supergravity fields in a D-brane are regarded as well [63–65]. Hence, in order to solve analytically the equations (11) and (12) we apply the redefinition

$$u = Ax + Bt. \quad (13)$$

Thus, the fields ϕ and χ take the form

$$\phi(x, t) \mapsto \phi(u), \quad (14)$$

$$\chi(x, t) \mapsto \chi(u). \quad (15)$$

Therefore, Eqs. (11) and (12) are respectively led to the following expressions:

$$\begin{aligned} [B^2 \eta_1^{00} + A^2 \eta_1^{11} + AB(\eta_1^{10} + \eta_1^{01})] \phi_{uu} + [B^2 K^{00} + A^2 K^{11} + AB(K^{10} + K^{01})] \chi_{uu} \\ + [BH_\phi^0 + AH_\phi^1 - BH_\phi^0 - AH_\phi^1] \phi_u + [BH_\chi^0 + AH_\chi^1 - BJ_\phi^0 - AJ_\phi^1] \chi_u + V_\phi = 0, \end{aligned} \quad (16)$$

and

$$\begin{aligned} [B^2 K^{00} + A^2 K^{11} + AB(K^{10} + K^{01})] \phi_{uu} + [B^2 \eta_2^{00} + A^2 \eta_2^{11} + AB(\eta_2^{10} + \eta_2^{01})] \chi_{uu} \\ + [BJ_\phi^0 + AJ_\phi^1 - BH_\chi^0 - AH_\chi^1] \phi_u + [BJ_\chi^0 + AJ_\chi^1 - BJ_\chi^0 - AJ_\chi^1] \chi_u + V_\chi = 0. \end{aligned} \quad (17)$$

Such system of coupled equations can be simplified and further computed by naming

$$\alpha := -B^2\eta_1^{00} - A^2\eta_1^{11} - AB(\eta_1^{10} + \eta_1^{01}), \quad (18)$$

$$\beta := B^2 K^{00} + A^2 K^{11} + AB(K^{10} + K^{01}), \quad (19)$$

$$\gamma := -B^2\eta_2^{00} - A^2\eta_2^{11} + AB(\eta_2^{10} + \eta_2^{01}), \quad (20)$$

$$L := BH_\phi^0 + AH_\phi^1 - BH_\phi^0 - AH_\phi^1, \quad (21)$$

$$S := BH_\chi^0 + AH_\chi^1 - BJ_\phi^0 - AJ_\phi^1, \quad (22)$$

Hence by denoting $A_u \equiv \partial A / \partial u$ for any quantity A , the system (16) and (17) can be expressed forthwith as:

$$-\alpha\phi_{uu} + \beta\chi_{uu} + S\chi_u + V_\phi = 0, \quad (23)$$

$$\beta\phi_{uu} - \gamma\chi_{uu} - S\phi_u + V_\chi = 0, \quad (24)$$

which can be led to

$$-\alpha\phi_u\phi_{uu} + \beta\phi_u\chi_{uu} + S\phi_u\chi_u + \phi_u V_\phi = 0, \quad (25)$$

$$\beta\chi_u\phi_{uu} - \gamma\chi_u\chi_{uu} - S\phi_u\chi_u + \chi_u V_\chi = 0, \quad (26)$$

whose sum reads:

$$-\alpha\phi_u\phi_{uu} - \gamma\chi_u\chi_{uu} + \beta(\phi_u\chi_{uu} + \chi_u\phi_{uu}) + \phi_u V_\phi + \chi_u V_\chi = 0. \quad (27)$$

It implies that

$$-\frac{\alpha}{2}\phi_u^2 - \frac{\gamma}{2}\chi_u^2 + \beta\phi_u\chi_u + V(\phi, \chi) = E_0, \quad (28)$$

where E_0 is a constant of integration which can be lead to be zero, in order to get solitonic solutions.

Now, the rescaling

$$\phi(u) \mapsto \sqrt{\alpha}\phi(u) := \tilde{\phi}(u), \quad (29)$$

$$\chi(u) \mapsto \sqrt{\gamma}\chi(u) := \tilde{\chi}(u), \quad (30)$$

makes Eq.(31) to be written as

$$-\frac{\tilde{\phi}_u^2}{2} - \frac{\tilde{\chi}}{2} + \frac{\beta}{\alpha\gamma}\tilde{\phi}_u\tilde{\chi}_u + V(\tilde{\phi}, \tilde{\chi}) = 0. \quad (31)$$

By rotating the variables $\tilde{\phi}$ and $\tilde{\chi}$

$$\begin{pmatrix} \tilde{\phi} \\ \tilde{\chi} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \theta \\ \zeta \end{pmatrix}, \quad (32)$$

and subsequently rescaling the new variables as

$$\theta = \sqrt{\frac{2}{1+\tilde{\beta}}} \sigma, \quad \zeta = \sqrt{\frac{2}{1-\tilde{\beta}}} \xi, \quad (33)$$

we finally arrived at the following expression:

$$-\frac{\sigma_u^2}{2} - \frac{\xi_u}{2} + V(\sigma, \xi) = 0. \quad (34)$$

It is worth to emphasize that the parameters α and γ must be greater than zero, and it is necessary to impose for $\alpha > 0$ that

$$B^2 \eta_1^{00} + A^2 \eta_1^{11} + AB(\eta_1^{10} + \eta_1^{01}) < 0, \quad (35)$$

$$B^2 \eta_2^{00} + A^2 \eta_2^{11} + AB(\eta_2^{10} + \eta_2^{01}) < 0. \quad (36)$$

The parameters η_1^{00} and η_2^{00} can be thus restricted by the other parameters that provide the Lorentz violation, accordingly:

$$\eta_1^{00} < -\frac{A^2 \eta_1^{11} + AB(\eta_1^{10} + \eta_1^{01})}{B^2}, \quad (37)$$

$$\eta_2^{00} < -\frac{A^2 \eta_2^{11} + AB(\eta_2^{10} + \eta_2^{01})}{B^2}. \quad (38)$$

The potential $V(\phi, \chi)$ is supposed to be provided in terms of the superpotential $W(\phi, \chi)$, by

$$V(\sigma, \xi) = \frac{1}{2} W_\sigma^2 + \frac{1}{2} W_\xi^2, \quad (39)$$

where $W_\xi = \partial W / \partial \xi$ and $W_\sigma = \partial W / \partial \sigma$. Notice that the critical points of the superpotential $W(\sigma, \xi)$ provide the set of vacua $\{(\sigma, \xi) \in \mathbb{R}^2 : V(\sigma, \xi) = 0\}$ for the field theory model that is regarded. The energy density has the form

$$\epsilon(x) = \frac{1}{2} (\sigma'^2 + \xi'^2 + W_\sigma^2 + W_\xi^2) = \frac{1}{2} [(\sigma' - W_\sigma)^2 + (\xi' - W_\xi)^2] + dW. \quad (40)$$

The minimum energy solutions thus obey the expressions

$$\sigma_u = \pm W_\sigma, \quad \xi_u = \pm W_\xi, \quad (41)$$

leading us to the BPS energy [46]

$$E_{\text{BPS}} = |W(\sigma(\infty), \xi(\infty)) - W(\sigma(-\infty), \xi(-\infty))|, \quad (42)$$

for smooth superpotentials. In terms of the superpotential, the equations of motion for static fields read

$$\sigma'' = W_\sigma W_{\sigma\sigma} + W_\xi W_{\xi\sigma}, \quad (43)$$

$$\xi'' = W_\sigma W_{\sigma\xi} + W_\xi W_{\xi\xi}, \quad (44)$$

which are solved by the first order equations (41), for $W_{\sigma\xi} = W_{\xi\sigma}$. Solutions to these first order equations are well known to be BPS states, which solve the equations of motion. The sectors where the potential has BPS states are named BPS sectors.

As an example, let us consider the model characterized by the superpotential

$$W(\sigma, \xi) = -\lambda\sigma + \frac{\lambda}{3}\sigma^3 - \mu\sigma\xi^2, \quad (45)$$

where λ and μ are real positive dimensionless coupling constants. The superpotential (45) has been studied by Shifman and Voloshin in the framework of $N = 1$ supersymmetric Wess-Zumino models with two chiral superfields [66, 67]. In the purely bosonic framework the presence of domain walls and its stability has been analyzed in the references [46–48]. Moreover, in [68, 69] the complete structure of this type of solutions is given in two critical values of the coupling between the two scalar fields, by exploiting the integrability of the analogue mechanical system associated with this model. This model, the so called BNRT model, has been further employed in several systems in the context of field theory and condensed matter [47].

For the superpotential (45), the associated potential is provided by

$$V(\sigma, \xi) = \frac{1}{2} \left[\lambda^2 + \lambda^2 \sigma^2 (\sigma^2 - 2) + \mu^2 \xi^2 \left(\xi^2 - \frac{2\lambda}{\mu} \right) + 2\mu^2 \left(\frac{\lambda}{\mu} + 2 \right) \sigma^2 \xi^2 \right]. \quad (46)$$

For $\lambda/\mu > 0$, the model presents four supersymmetric minima (σ, ξ) , given by:

$$(\pm 1, 0) \quad \text{and} \quad \left(0, \sqrt{\frac{\lambda}{\mu}} \right). \quad (47)$$

Now, from the set (41) the following expression can be derived:

$$\frac{d\sigma}{d\xi} = \frac{W_\sigma}{W_\xi} = \frac{\lambda(\sigma^2 - 1) + \mu\xi^2}{2\mu\sigma\xi}. \quad (48)$$

Hence the following first order differential equation is immediately derived:

$$\frac{d\xi}{du} = \begin{cases} \pm 2\mu\xi\sqrt{1 + c_0\xi^{\lambda/\mu} - \frac{\mu}{\lambda-2\mu}\xi^2}, & \lambda \neq 2\mu, \\ \pm 2\mu\xi\sqrt{1 + \xi^2(\ln \xi + c_1)}, & \lambda = 2\mu, \end{cases} \quad (49)$$

for c_0 and c_1 constants of integration. These equations have analytical solutions. First, Eq.(49) presents solutions, for $c_0 < -2$ and $\lambda = \mu$:

$$\xi^{(1)}(u) = \frac{2}{\sqrt{c_0^2 - 4} \cosh(2\mu u) - c_0}, \quad (50)$$

$$\sigma^{(1)}(u) = \frac{\sqrt{c_0^2 - 4} \sinh(2\mu u)}{\sqrt{c_0^2 - 4} \cosh(2\mu u) - c_0}. \quad (51)$$

For $c_0 < 1/16$ and $\lambda = 4\mu$:

$$\xi^{(2)}(u) = -\frac{2}{\sqrt{\sqrt{1 - 16c_0} \cosh(4\mu u) + 1}}, \quad (52)$$

$$\sigma^{(2)}(u) = \frac{\sqrt{1 - 16c_0} \sinh(4\mu u)}{\sqrt{1 - 16c_0} \cosh(4\mu u) + 1}. \quad (53)$$

These solutions for the corresponding original fields ϕ and χ thus read:

$$\phi^{(j)}(u) = \frac{1}{\sqrt{\alpha}} \left(\frac{\sigma^{(j)}(u)}{\sqrt{1 + \tilde{\beta}}} - \frac{\xi^{(j)}(u)}{\sqrt{1 - \tilde{\beta}}} \right), \quad (54)$$

$$\chi^{(j)}(u) = \frac{1}{\sqrt{\gamma}} \left(\frac{\sigma^{(j)}(u)}{\sqrt{1 + \tilde{\beta}}} + \frac{\xi^{(j)}(u)}{\sqrt{1 - \tilde{\beta}}} \right), \quad (55)$$

for $j = 1, 2$.

The profile for the fields $\phi^{(j)}(u)$ and $\chi^{(j)}(u)$ are depicted in Fig. 1, which shows the influence of the Lorentz violation on the field configurations. Furthermore, in Fig. 2 we can see the orbits connecting the vacua. In the next section we shall describe the so-called configurational entropy (CE). In this case, similarly to the seminal result by Gleiser and Stamatopoulos (GS) [35], we are going to postulate the travelling configurational entropy, that can be employed in order to analyze the entropic profile of any localized configuration of fields. Besides, it can be further used in classical field theories presenting solutions in the travelling variables.

3. TRAVELLING CONFIGURATIONAL ENTROPY (TCE)

Recently GS showed that scalar field configurations, spatially localized and with finite energy, presenting the same energy can be discriminated via the so-called configurational entropy [35]. Analogously to the Shannon's information theory, the configurational entropy can be described by the expression

$$S_c[f] = - \int d^d \vec{k} \, \tilde{f}(\vec{k}) \ln[\tilde{f}(\vec{k})], \quad (56)$$

where d denotes the number of space dimensions, $\tilde{f}(\vec{k}) := f(\vec{k})/f_{\max}(\vec{k})$, and $f(\vec{k})$ is defined as the modal fraction

$$f(\vec{k}) = \frac{|F(\vec{k})|^2}{\int d^d \vec{k} \, |F(\vec{k})|^2}. \quad (57)$$

The quantity $f_{\max}(\vec{k})$ denotes the maximal modal fraction, namely, the mode the contributes to the maximal contribution, and $F(\vec{k})$ was defined in [35] as the Fourier transform of the energy density. The higher the configurational entropy the higher the energy of the solutions, corresponding to the most ordered solutions [39]. The configurational entropy is moreover responsible to point out which solution is the most ordered one among a family of infinite degenerated solutions. Hereon we likewise propose that the function $F(\vec{k})$ also represents the Fourier transform of the energy density as well. Notwithstanding, the variable of integration is the travelling variable, namely

$$F_T[\vec{k}] = \frac{1}{(2\pi)^{d/2}} \int d^d u \, e^{i\vec{k}\vec{u}} T^{00}(\vec{u}). \quad (58)$$

Moreover, from the Plancherel theorem it follows that

$$\int d^d u \, |F(\vec{k})|^2 = \int d^d u \, |T^{00}(\vec{u})|^2. \quad (59)$$

In this context, the modal fraction obeys the same relation (57) and the configurational entropy can be determined by Eq.(56). Thus we can achieve the entropic measure of localized scalar configurations that present their structure determined by the travelling variables. In this case, we name the expression (56) as the Travelling Configurational Entropy (TCE). Moreover, our framework can be straightforwardly led to the results in [35], when the limit $B \rightarrow 0$ is taken in Eq.(13). The description heretofore presented can be hence applied to Lorentz violation models in order to analyze the entropic profile of travelling solitons. It is

moreover worth to mention that the travelling-like solutions can be recovered by adjusting the Lorentz violation parameters and leading to the usual Lorentz symmetry.

In order to analyze the energy for the obtained configurations, the energy-momentum tensor

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \chi)} \partial^\nu \chi - g^{\mu\nu} \mathcal{L}, \quad (60)$$

is now regarded for the Lagrangian density:

$$T^{00}(u) = \frac{1}{2}(\eta_1^{00} B^2 + \eta_1^{11} A^2) \phi_u^2 + \frac{1}{2}(\eta_2^{00} B^2 + \eta_2^{11} A^2) \chi_u^2 \quad (61)$$

$$+ (K^{00} B^2 + K^{11} A^2) \chi_u \phi_u - A(H^1 \phi_u + J^1 \chi_u) + V(\phi, \chi). \quad (62)$$

The profile of the energy density is presented in Fig. 3. Besides Eqs.(37) and (38), the constraint $-1 < \tilde{\beta} < 1$ holds, implying hence the following constraints for K^{00} :

$$-\frac{1}{B^2} \left[\frac{\sqrt{\alpha\gamma}}{2} + A^2 K^{11} + (K^{10} + K^{01})AB \right] < K^{00} < \frac{1}{B^2} \left[\frac{\sqrt{\alpha\gamma}}{2} - A^2 K^{11} - (K^{10} + K^{01})AB \right].$$

In (1+1) dimensions, Eq.(58) obviously reads

$$F[k] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} du e^{iku} T^{00}(u). \quad (63)$$

Now, by using Eq.(39) we get the following transform:

$$F[k] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} du e^{iku} \left[\frac{2b_+ + 1}{2} \sigma_u^2 + \frac{2b_- + 1}{2} \xi_u^2 + b_3 \sigma_u \xi_u + b_4 \sigma_u + b_5 \xi_u \right], \quad (64)$$

where

$$b_{\pm} := \frac{a_1^2}{2} \left(\frac{\gamma_1}{\alpha} + \frac{\gamma_2}{\gamma} \pm \frac{2\gamma_3}{\sqrt{\alpha\gamma}} \right), \quad b_3 := a_1 a_2 \left(\frac{\gamma_1}{\alpha} + \frac{\gamma_2}{\gamma} \right), \quad (65)$$

$$b_4 := -a_1 \left(\frac{\gamma_4}{\sqrt{\alpha}} + \frac{\gamma_5}{\sqrt{\gamma}} \right), \quad b_5 := a_2 \left(\frac{\gamma_4}{\sqrt{\alpha}} - \frac{\gamma_5}{\sqrt{\gamma}} \right), \quad (66)$$

and

$$\begin{aligned} a_1 &= (1 + \tilde{\beta})^{-1/2}, \quad a_2 = (1 - \tilde{\beta})^{-1/2}, \\ \gamma_1 &= \eta_1^{00} B^2 + \eta_1^{11} A^2, \quad \gamma_2 = \eta_2^{00} B^2 + \eta_2^{11} A^2, \\ \gamma_3 &= K^{00} B^2 + K^{11} A^2, \quad \gamma_4 = A H^1, \quad \gamma_5 = A J^1. \end{aligned}$$

Therefore the function $F[k]$ can be written as

$$F[k] = \sum_{\ell=1}^4 \sum_{m=1}^3 r_{(m)}(\ell) I^{(m)}(\ell), \quad (67)$$

where

$$I^{(1)}(\ell) = 2^\ell \sum_{n=1}^2 \tilde{I}_n(\ell), \quad (68)$$

$$I^{(2)}(\ell) = 2^\ell \sum_{n=1}^2 \sum_{j=1}^2 (-1)^{n-1} \tilde{I}_n^{(\ell+1)}(\ell), \quad (69)$$

$$I^{(3)}(\ell) = 2^{\ell+2} \sum_{n=1}^2 \sum_{j=1}^2 \bar{I}_n^{(j)}(\ell+3), \quad (70)$$

where $\bar{I}_n^{(j)}(\ell+3) = I_n^{(j)}(\ell+4)$. Besides, we have used the following notation:

$$r_{(1)}(\ell) = q_\ell, \quad r_{(2)}(\ell) = s_\ell, \quad r_{(3)}(\ell) = p_\ell, \quad (71)$$

$$q_1 = 2\mu g_4 c_0, \quad q_2 = -8\mu g_4, \quad q_3 = -8\mu g_4, \quad (72)$$

$$q_4 = 8\mu g_1 - 4\mu g_2(c_0^2 - 4), \quad s_1 = -2g_5\sqrt{c_0^2 - 4}, \quad s_2 = -4\mu g_3 c_0\sqrt{c_0^2 - 4}, \quad (73)$$

$$s_3 = 16\mu g_3\sqrt{c_0^2 - 4}, \quad p_1 = 4\mu g_2\sqrt{c_0^2 - 4}, \quad (74)$$

accordingly.

In Eqs.(68-70) it reads

$$\begin{aligned} \tilde{I}_n(\ell) &= \frac{1}{2\mu[C_0(B_0^2 - D_0^2)]^\ell} \frac{\Gamma\left[\ell + (-1)^{n-1} \frac{ik}{2\mu}\right]}{\Gamma\left[\ell + 1 + (-1)^{n-1} \frac{ik}{2\mu}\right]} \times \\ &{}_2F_1\left[\ell + (-1)^{n-1} \frac{ik}{2\mu}, \ell, \ell, \ell + 1 + (-1)^{n-1} \frac{ik}{2\mu}; X_1, Y_1\right], \end{aligned} \quad (75)$$

and

$$\begin{aligned} I_n(\ell) &= \frac{1}{2\mu[C_0(B_0^2 - D_0^2)]^\ell} \frac{\Gamma\left[(-1)^{j-1} \bar{A}/\bar{B} + \ell + (-1)^{n+j} ik/\bar{B}\right]}{\Gamma\left[(-1)^{j-1} \bar{A}/\bar{B} + \ell + 1 + (-1)^{n+j} ik/\bar{B}\right]} \times \\ &{}_2F_1\left[(-1)^{j-1} \frac{\bar{A}}{\bar{B}} + \ell + (-1)^{n+j} \frac{ik}{\bar{B}}, \ell, \ell, (-1)^{j-1} \frac{A}{B} + \ell + 1 + (-1)^{j+n} \frac{ik}{\bar{B}}; X_1, Y_1\right], \end{aligned} \quad (76)$$

where $\bar{A} = 2\mu$, $C_0 := \sqrt{c_0^2 - 4}$, $D_0 := \sqrt{c_0^2 - C_0^2}/C_0$, $B_0 := c_0/C_0$, $X_1 := 1/(B_0 + D_0)$, and $Y_1 := 1/(B_0 - D_0)$.

Moreover,

$$\bar{I}_n^{(j)}(\ell) = I_n^{(j)}(\ell+1), \quad (77)$$

with $A = 4\mu$ and $B = 2\mu$.

The fraction mode can be thus written in a more compact form:

$$f(k) = \frac{\sum_{\ell, \ell'=1}^4 \sum_{m, m'=1}^3 r_{(m)}(\ell) r_{(m')}(\ell')^* I^{(m)}(\ell) I^{(m')}(\ell')^*}{\sum_{\ell, \ell'=1}^4 \sum_{m, m'=1}^3 \int_{-\infty}^{\infty} dk r_{(m)}(\ell) r_{(m')}(\ell')^* I^{(m)}(\ell) I^{(m')}(\ell')^*}. \quad (78)$$

In Fig. 4 we can realize the behavior of the modal fraction and realize how its profile is influenced by the Lorentz violation parameters.

To compute the configurational entropy, we must integrate Eq. (56) numerically. The results are shown in Fig. 5, where the TCE is plotted as a function of the parameter k^{00} . From that figure we can check that there is a region of k^{00} where the existence of solutions is forbidden by entropy. In this case, the region of parameter travelling where the fields is most prominent are that given by values $k^{00} > -0.06$ with the corresponding TCE $S_c = 0$. Moreover, for $k^{00} = -0.06$ the field configurations undergo a kind of phase transition, where the two-kink solution in the fields $\phi^{(1)}(u)$ and $\chi^{(1)}(u)$ converges into a single kink. Another very important revelation that comes from entropy measure has risen when $k^{00} \rightarrow -0.055$. In this limit we have a symmetric distribution of energy density around the origin, showing that the field configurations are equally distributed in both the sides of vertical axes. Moreover, in that limit once more the configurations undergo a new transition in their structures, where the associated solutions converge in lumps configurations.

For the sake of completeness, we have examined how the results vary with respect to the fields $\phi^{(2)}(u)$ and $\chi^{(2)}(u)$, where we conclude that the qualitative features remain the same.

The above results lead us to conclude that the TCE can be used in other to extract a rich information about the structure of the configurations which is clearly related to their travelling profiles. Here, we found that the best ordering for the solutions are that given by $k^{00} \rightarrow -0.055$ where the configuration is symmetric around of origin.

5. CONCLUSIONS

In this work, following a recent work [35], we showed that it is possible to construct a configurational entropy measure in functional space from the field configurations where travelling wave solutions can be studied. Thus, we applied the approach to investigate the existence and properties of travelling solitons in Lorentz and *CPT* breaking scenarios for a class of models with two interacting scalar fields. Here, we obtained a complete set of exact solutions for the model studied, which display both double and single-kink configurations.

We have found that the so-called Configurational Entropy for travelling solitons, which we name as the Travelling Configurational Entropy (TCE), shows that the best value for the parameter responsible to break the Lorentz symmetry is the one which has energy density profile symmetric with respect to the origin. In this way, the information-theoretical measure of travelling solitons in Lorentz symmetry violation scenarios opens a new window to probe situations where the parameters responsible for breaking the symmetries are random. In this case, the TCE selects the best value. Moreover, the variable used in this work, $u = Ax + Bt$, when compared with the usual boosted variable, $u = \gamma(x + vt)$, allows that the parameters A and B can be chosen in a range larger than the corresponding ones in the boosted variable, allowing the appearance of superluminal solitons [19]. Thus, the TCE provides a complementary perspective to investigate the causality and superluminal behavior in classical field theories such as k-essence theories and MOND-like theories of gravity [70, 71]. Other applications where TCE can be used to relate the dynamical and informational content of physical system is found in the so-called Galilean field theories. In this context, we are presently interested in the possibility of constructing the entropic profile of Galileons on cosmological backgrounds [72].

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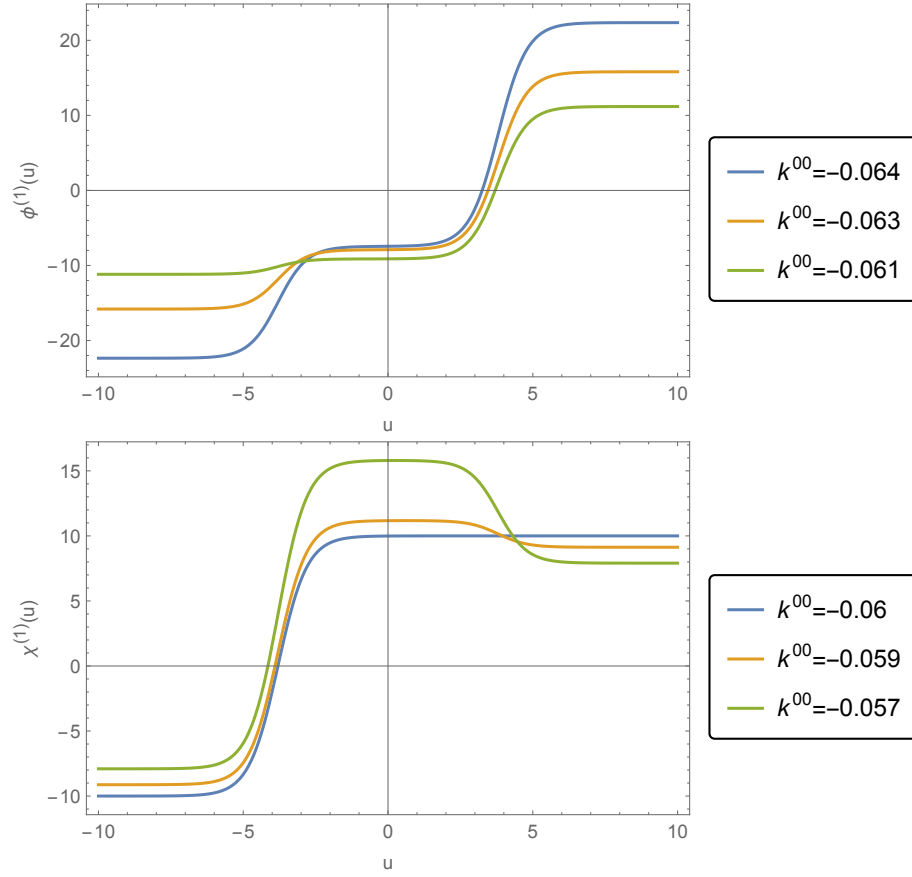


FIG. 1: Field configurations $\phi^{(1)}(u)$ and $\chi^{(1)}(u)$ for $A = B = 1$, $\mu = 1$, $\eta_1^{00} = \eta_2^{00} = -3.05$, $\eta_1^{11} = \eta_2^{11} = 1.01$, $\eta_1^{01} = \eta_2^{01} = 1.01$, $\eta_1^{10} = \eta_2^{10} = 1.02$, $k^{01} = 0.03$, $k^{10} = 0.02$, $k^{11} = 0.01$, and $c_0 = -2.000001$.

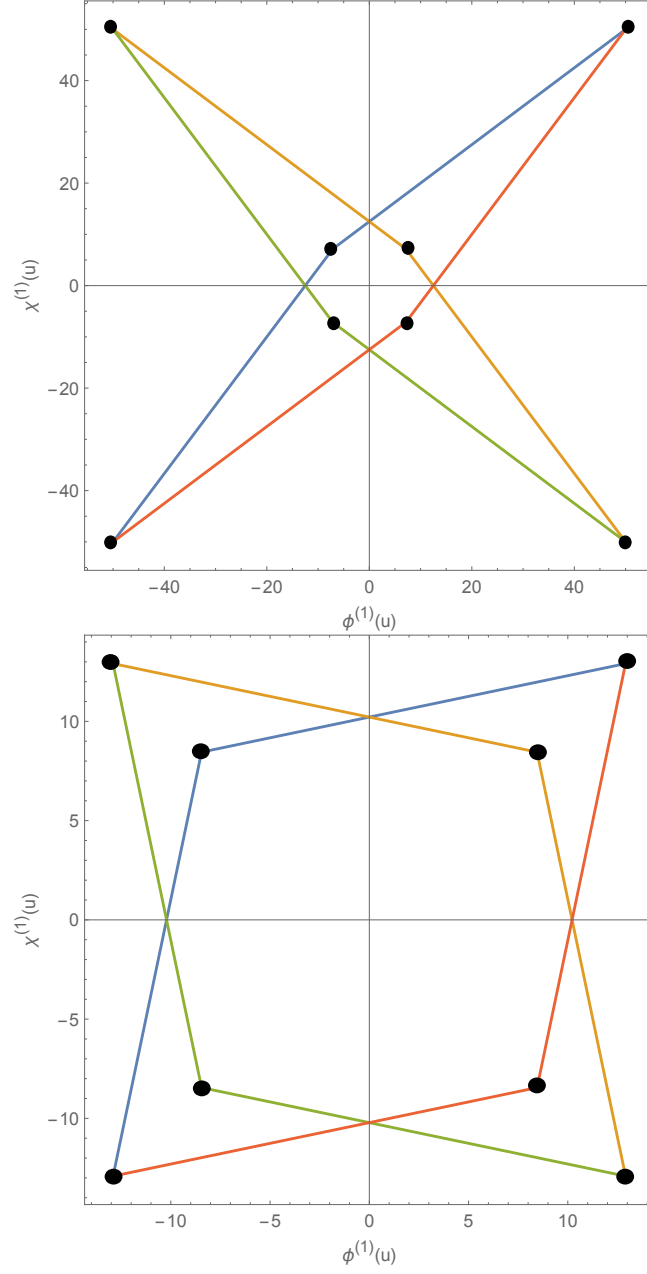


FIG. 2: Orbit for the solutions and vacuum states of the potential for $A = B = 1$, $\mu = 1$, $\eta_1^{00} = \eta_2^{00} = -3.05$, $\eta_1^{11} = \eta_2^{11} = 1.01$, $\eta_1^{01} = \eta_2^{01} = 1.01$, $\eta_1^{10} = \eta_2^{10} = 1.02$, $k^{01} = 0.03$, $k^{10} = 0.02$, $k^{11} = 0.01$, and $c_0 = -2.000001$. The plot on the top of the figure show the case with $k^{00} = -0.0648$ and the bottom ones with $k^{00} = -0.0620$.

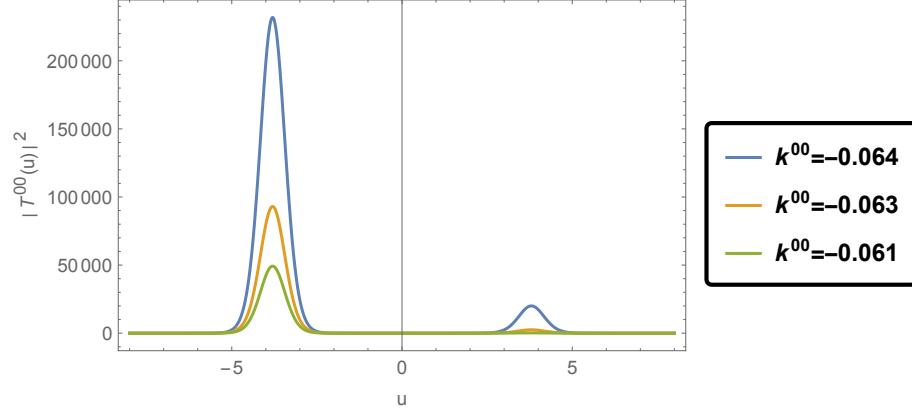


FIG. 3: Energy density for $A = B = 1$, $\mu = 1$, $\eta_1^{00} = \eta_2^{00} = -3.05$, $\eta_1^{11} = \eta_2^{11} = 1.01$, $\eta_1^{01} = \eta_2^{01} = 1.01$, $\eta_1^{10} = \eta_2^{10} = 1.02$, $k^{01} = 0.03$, $k^{10} = 0.02$, $k^{11} = 0.01$, and $c_0 = -2.000001$.

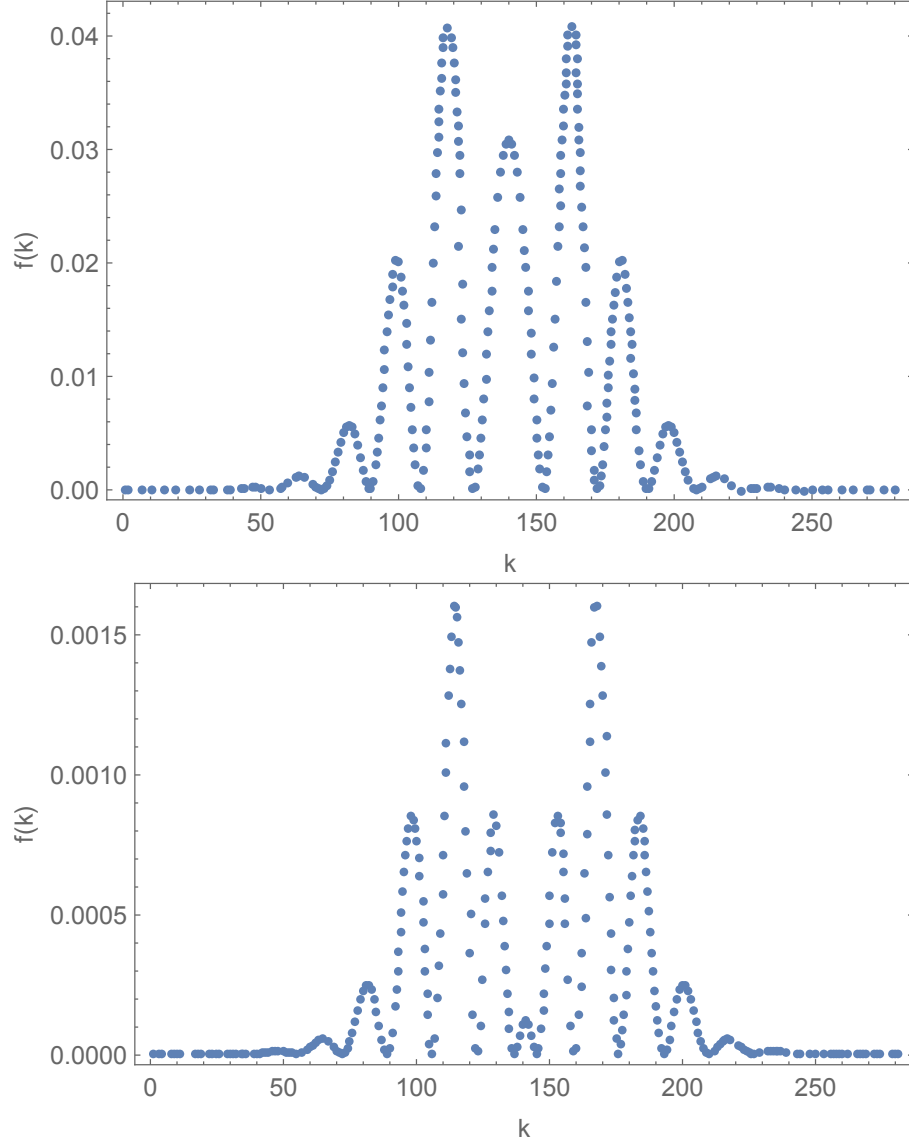


FIG. 4: Modal fractions for $A = B = 1$, $\mu = 1$, $\eta_1^{00} = \eta_2^{00} = -3.05$, $\eta_1^{11} = \eta_2^{11} = 1.01$, $\eta_1^{01} = \eta_2^{01} = 1.01$, $\eta_1^{10} = \eta_2^{10} = 1.02$, $k^{01} = 0.03$, $k^{10} = 0.02$, $k^{11} = 0.01$, and $c_0 = -2.000001$. The plot on the top of the figure show the case with $k^{00} = -0.058$ and the bottom ones with $k^{00} = -0.056$.

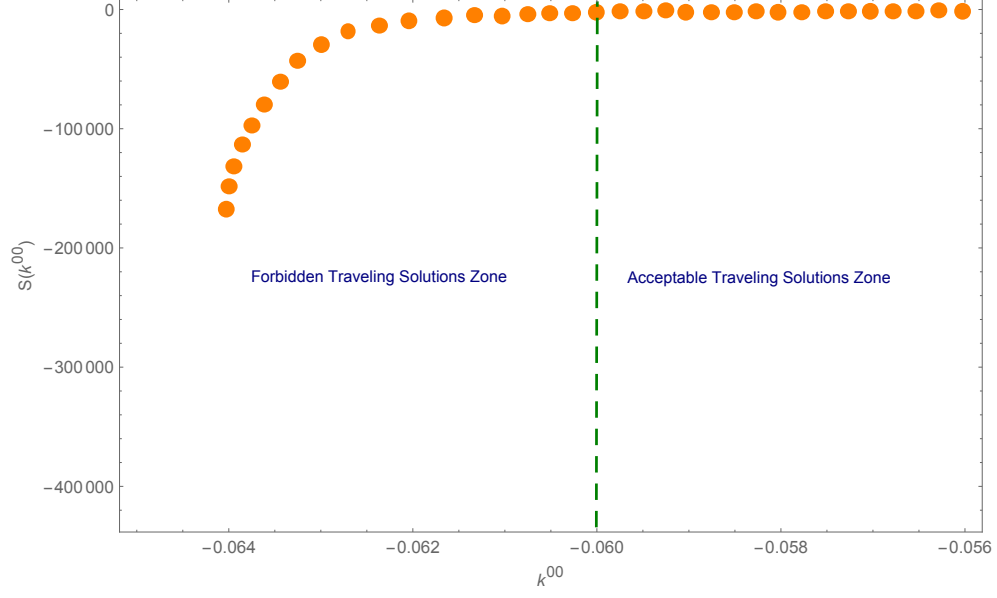


FIG. 5: travelling configurational entropy for $A = B = 1$, $\mu = 1$, $\eta_1^{00} = \eta_2^{00} = -3.05$, $\eta_1^{11} = \eta_2^{11} = 1.01$, $\eta_1^{01} = \eta_2^{01} = 1.01$, $\eta_1^{10} = \eta_2^{10} = 1.02$, $k^{01} = 0.03$, $k^{10} = 0.02$, $k^{11} = 0.01$, and $c_0 = -2.000001$.